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J. M. ZIMMERMAN

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I. INTRODUCTION

In this paper we shall investigate solutions of equations of the form

$$a(y)u_{tt} = b(y)u_{xx} + u_{yy} \quad , \tag{1}$$

which have continuous derivatives of the second order, for initial data given on the timelike manifold $y = 0$:

$$\begin{aligned} u(t,x,0) &= f(t,x) \\ u_y(t,x,0) &= g(t,x) \quad . \end{aligned} \tag{2}$$

It is assumed that $a(y) \in c^2$, $b(y) \in c^0$ and $a(y) > 0$ for y in some finite interval $0 \leq y \leq h$.

When $b > 0$ over at least part of this interval, such problems are "improper" in the sense of Hadamard. That is, even under conditions on f and g which insure the existence of a unique solution u to (1),(2) there is, in general, no suitable norm (involving derivatives up to a fixed finite order) in which u depends continuously on the initial data f,g. For some improper problems it is possible to obtain continuous dependence by restricting attention to solutions which possess a fixed finite bound in a suitably chosen norm. This approach has been extensively studied by F. John[1], and in certain situations he has shown that the resulting continuous dependence is of the Hölder type. That is, if u and u' are solutions corresponding to initial data h, h', respectively, then,

(3) $$|u-u'| < M\delta^{\theta}$$

when $|h-h'| < \delta$. Here M and θ are positive constants ($0 < \theta \leq 1$), and the norm "$|\ |$" is appropriately chosen (involving u and at most a finite number of its derivatives).

The improper problem (1), (2) is, however, not of this class. In fact John has shown[1] explicitly for the wave equation that even prescribing a bound M for u and a finite number of its derivatives results in only a weaker type of logarithmic continuous dependence described by

(4) $$|u-u'| \leq M\ (\log \delta^{-1})^{-\theta} \quad .$$

In this report we shall find a solution to (1), (2) whose dependence on the initial data f,g is of the form (3)(in fact with $\theta = 1$). In order to accomplish this certain restrictions will be placed on the data f,g.

Suppose first that $f,g \in L_2(-\infty,\infty)$ with respect to both x and t (we write $f,g \in L_2^{t,x}$). Let the Fourier transforms of f and g with rexpect to x be denoted by $\overline{F}(t,\omega)$ and $\overline{G}(t,\omega)$, respectively, and suppose the transforms of $\overline{F}$ and $\overline{G}$ with respect to t are given by $F(\tau,\omega)$, $G(\tau,\omega)$. We have already shown (the problem (1), (2) is a special case of the nonlinear Cauchy problems treated in [2])that if F and G have compact supports as functions of both ω and τ, then the initial value problem (1), (2) has a unique $L_2^{t,x}$ solution $u(t,x,y)$ for $0 \leq y \leq h$. Moreover, u has Fourier transforms with respect to x and t which have compact support and the continuous dependence of u on the initial data f,g is of the type (3) with $\theta = 1$.

We shall now show that for the problem (1),(2) it is sufficient to assume only that $\overline{F}(t,\omega)$, $\overline{G}(t,\omega)$ have compact support as functions of ω for all t, and $\tau^2 F(\tau,\omega)$, $\tau^2 G(\tau,\omega) \in L_2^\tau$ for all ω in this support in order to prove the existence of unique C^2 solutions which depend Lipschitz continuously on the initial data.

Our methods are applicable to more general equations than (1). In fact entirely analogous results can be obtained for the ultra-hyperbolic equation

$$(5) \quad \sum_{i=1}^{m} a_i(y) u_{t_i t_i} = \sum_{j=1}^{n} b_j(y) u_{x_j x_j} + u_{yy} \quad ,$$

where $a_i(y) \in C^2$, $b_j(y) \in C^0$, $a_i > 0$ for $i = 1,\ldots,m$, $j = 1,\ldots,n$, and the initial data is given on the timelike manifold $y = 0$. To keep the notation simple here we shall explicitly treat only the case $m = 1$, $n = 1$ (i.e., the system (1),(2).

II. THE ASSOCIATED "REDUCED" ORDINARY DIFFERENTIAL EQUATION

Let the region $R(h) \subset \mathbb{R}^3$ be defined by

$$R(h) = \{(t,x,y): -\infty < t,\ x < \infty,\ 0 \le y \le h\} ,$$

where h is arbitrary but finite. For $(t,x,y) \in R(h)$ we define $\overline{F},\overline{G},F,G$ as in Section I, and formally define $\overline{U}(t,\omega,y)$, $U(\tau,\omega,y)$ as the Fourier transforms of $u(t,x,y)$ and $\overline{U}(t,\omega,y)$ with respect to x and t respectively. We define classes of functions $B(x,\omega_o)$, $\overline{\mathfrak{B}}(\omega,\omega_o)$ and $\mathcal{B}(\omega,\omega_o)$ as follows:

The function $f(t,x) \in B(x,\omega_o)$ if

(a) $f(t,x) \in L_2^{t,x}$. (Therefore $\overline{F}(t,\omega) \in L_2^\omega$ for each t and $F(\tau,\omega) \in L_2^{\tau,\omega}$).

(b) $\overline{F}(t,\omega)$ has support contained in the interval $|\omega| \leq \omega_0$ for all t.

(c) $\tau^2 F(\tau,\omega) \in L_2^\tau$ for all $|\omega| \leq \omega_0$.

(Such functions f can be referred to as "x-band-limited" with band-limit ω_0). It follows from (b) that $F(\tau,\omega)$ also vanishes for $|\omega| > \omega_0$ and all τ. The classes of such functions $\overline{F}(t,\omega)$ and $F(\tau,\omega)$ will be denoted by $\overline{\mathcal{B}}(\omega,\omega_0)$ and $\mathcal{B}(\omega,\omega_0)$, respectively. The function $u(t,x,y)$ will be said to belong to $B(x,\omega_0,h)$ if $u(t,x,y) \in B(x,\omega_0)$ for y satisfying $0 \leq y \leq h$.

Our aim is to prove the existence in $R(h)$ of a unique $B(x,\omega_0,h)$ solution to (1),(2) which is twice continuously differentiable in all variables. In order to do this we first formally reduce the problem by the methods of Fourier transforms to the study of an ordinary differential equation. The properties of the solution to this latter equation permit the formal procedure to be rigorously justified. The existence, uniqueness and continuous dependence of solutions to the original problem (1),(2) can then be established.

From (1),(2) we have immediately the reduced ordinary differential equation

$$(6) \quad \frac{d^2U(\tau,\omega,y)}{dy^2} + [a(y)\tau^2 - b(y)\omega^2]U(\tau,\omega,y) = 0$$

with initial data

$$U(\tau,\omega,0) = F(\tau,\omega)$$

$$(7) \quad \frac{dU(\tau,\omega,0)}{dy} = G(\tau,\omega)$$

in which we think of τ and ω as parameters. For each fixed

τ and ω, the classical existence theorems of ordinary differential equations provide a unique solution $U(\tau,\omega,y)$ to this problem which formally on Fourier inversion yields the desired solution $u(t,x,y)$.

In order to justify this procedure and describe the properties of u, we first obtain an elementary estimate for the solution U to (6),(7).

Lemma 1: Fo $0 \leq y \leq h$ define constants A, C, D, E by

$$A = \max[a(y)]^{-1/4} \quad , \quad E = \max[a(y)]^{1/2}$$

$$C = \max \left|\frac{b(y)}{a(y)}\right| \quad , \quad D = \max \left|\frac{5[a'(y)]^2 - 4a(y)a''(y)}{16[a(y)]^3}\right| \quad .$$

Then the solution $U(\tau,\omega,y)$ to (6),(7) satisfies the inequality.

$$(8) \quad |U(\tau,\omega,y)| \leq A\left\{|\phi(\tau,\omega)| + Eh|\psi(\tau,\omega)|\right\} \exp\left\{h^2E^2(C\omega^2+D)\right\}$$

uniformly for $0 \leq y \leq h$, where

$$\phi(\tau,\omega) = [a(0)]^{1/4}F(\tau,\omega)$$

$$\psi(\tau,\omega) = [a(0)]^{-1/4}G(\tau,\omega) + \tfrac{1}{4}[a(0)]^{-3/4}a'(0)F(\tau,\omega)$$

To prove the lemma we consider first the case $\omega \neq 0$ and write (6) in the form

$$(9) \quad [\omega^2 a(y)]^{-1}U'' + \left[\frac{\tau^2}{\omega^2} - \frac{b(y)}{a(y)}\right] U = 0$$

Let $\lambda = \frac{\tau^2}{\omega^2}$, and make the changes of independent and dependent variables:

$$(10) \quad \begin{aligned} Z(\tau,\omega,y) &= \omega^{-1/2}[a(y)]^{1/4}U(\tau,\omega,y) \\ s &= \omega\int_0^y [a(w)]^{1/2}dw \quad . \end{aligned}$$

Corresponding to the original y interval $0 \leq y \leq h$, we have the s-interval.

$$(11)\quad -S(\omega) \equiv -|\omega|\int_0^h [a(w)]^{1/2} dw \leq s(\omega)$$

$$\leq |\omega|\int_0^h [a(w)]^{1/2} dw \equiv S(\omega)$$

which for $|\omega| \leq \omega_o$ is also finite. With these changes (7) and (9) take the form

$$Z(\tau,\omega,0) = \omega^{-1/2}[a(0)]^{1/4}F(\tau,\omega) = \phi(\tau,\omega)\omega^{-1/2}$$

$$(12)\quad \frac{dZ(\tau,\omega,0)}{ds} = \omega^{-3/2}\left\{[a(0)]^{-1/4}G(\tau,\omega) + \frac{1}{4}[a'(0)]^{-3/4}a'(0)F(\tau,\omega)\right\} \equiv \psi(\tau,\omega)\omega^{-3/2}$$

and

$$(13)\quad \frac{d^2Z(\tau,\omega,s)}{ds^2} + (\lambda-r(s))Z(\tau,\omega,s) = 0,$$

respectively, where

$$(14)\quad r(s) = \left[\frac{b(y)}{a(y)} - \frac{5/4(a'(y))^2-a(y)a''(y)}{4\omega^2(a(y))^3}\right]_{y=y(s)}$$

and primes denote differentiation with respect to y. It is now a simple matter to study the growth of Z as a function of the parameter λ. Integrating (13) by parts twice we have immediately that $Z(\tau,\omega,s)$ satisfies the integral equation

$$Z(\tau,\omega,s) = \omega^{-1/2}\phi(\tau,s)\cos\sqrt{\lambda}\, s + \omega^{-3/2}\psi(\tau,\omega)\,\frac{\sin\sqrt{\lambda}\, s}{\sqrt{\lambda}}$$

$$+ \frac{1}{\sqrt{\lambda}}\int_0^s \sin\sqrt{\lambda}\,(s-s')r(s')Z(\tau,\omega,s')ds' \quad .$$

From this expression it follows that for all λ

$$|Z(\tau,\omega,s)| \leq |\omega^{-1/2}||\phi(\tau,\omega)| + |s||\omega^{-3/2}||\psi(\tau,\omega)|$$

$$+ \int_0^{|s|} |s-s'||r(s')||Z(\tau,\omega,s')|ds' \quad ,$$

so that

$$|Z(\tau,\omega,s)| \leq \left\{|\omega^{-1/2}||\phi(\tau,\omega)| + |s||\omega^{-3/2}||\psi(\tau,\omega)|\right\}$$

$$(15) \qquad \exp\left\{|s| \int_0^{|s|} |r(s')|ds'\right\}$$

$$\leq \left\{|\omega^{-1/2}||\phi| + S(\omega)|\omega^{-3/2}||\psi|\right\}$$

$$\exp\left\{S(\omega) \int_0^{S(\omega)} |r(s')|ds'\right\} \ .$$

For large λ it is possible to obtain much sharper estimates for $|Z|$, but (15) is sufficient for our present purposes.

To complete the proof we use (10) and (15) to obtain estimates for $|U(\tau,\omega,y)|$. Consider the integral

$$\rho(s) = \int_0^{|s|} |r(s')|ds' \quad .$$

We have from (11) and (14)(using the constants defined above) that

$$|\rho(s)| \leq \int_0^{|\omega|Eh} |r(s')|ds' \leq \frac{Eh}{|\omega|} (C\omega^2+D)$$

uniformly in s. Thus from (10), (12) and (15) we have for all τ, $0 \leq y \leq h$, and each ω that

$$|U(\tau,\omega,y)| \leq |\omega|^{1/2}|a(y)|^{-1/4}|Z(\tau,\omega,y)|$$

$$\leq A\left\{|\phi(\tau,\omega)| + Eh\ |\psi(\tau,\omega)|\right\} \exp\left\{h^2E^2(C\omega^2+D)\right\} .$$

Since the case $\omega = 0$ is formally equivalent to the case $\omega \equiv 1$, $b \equiv 0$, the proof of the lemma is complete.

III. EXISTENCE AND CONTINUOUS DEPENDENCE OF THE SOLUTION

Having outlined a formal procedure for finding a solution u to (1),(2) we now define $u(t,x,y)$ by

$$(16) \qquad u(t,x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int U(\tau,\omega,y)e^{i(x\omega+t\tau)}d\tau d\omega \quad ,$$

where $U(\tau,\omega,y)$ is the solution to the ordinary differential equation (6),(7), and use Lemma 1 to prove the following

Theorem 1. If $f(t,x),g(t,x) \in B(x,\omega_o)$, then there exists a $C^2(t,x,y)$ solution $u(t,x,y)$ to the timelike initial value problem (1),(2) in $R(h)$ having the following properties:

(a) $u(t,x,y) \in B(x,\omega_o,h)(\overline{U}(t,\omega,y) \in \overline{\mathcal{B}}(\omega,\omega_o,h)$ and $U(\tau,\omega,y) \in \mathcal{B}(\omega,\omega_o,h))$ in $R(h)$. Actually for each fixed t and y in $R(h)$ we have even more: $u(t,x,y)$ is an entire analytic function of x of exponential type ω_o.

(b) $u(t,x,y)$ depends Lipschitz continuously on $f(t,x)$, $g(t,x)$ in the $L_2^{t,x}$ norm for all y satisfying $0 \leq y \leq h$.*

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* For any $B(x,\omega_o,h)$ function $h(t,x,y)$ we have immediately the inequalities

$$\left|\frac{\partial^{(n)}h(t,x,y)}{\partial x^n}\right| \leq \frac{\sqrt{\omega_o}}{\sqrt{\pi}}\left\{\int_{-\infty}^{\infty}\left|\frac{\partial^{(n)}h(t,x,y)}{\partial x^n}\right|^2 dx\right\}^{1/2}$$

$$\int_{-\infty}^{\infty}\left|\frac{\partial^{(n)}h(t,x,y)}{\partial x^n}\right|^2 dx \leq \omega_o^n\int_{-\infty}^{\infty}|h(t,x,y)|^2 dx.$$

(c) The function $u(t,x,y)$ is unique in the class of $B(x,\omega_o,h)$ functions which are also in $C^2(y)$.

Proof: For any function $v(t,x,y) \varepsilon L_2^{t,x}$ for each y we employ the notation

$$\int_{-\infty}^{\infty}\int |v(t,x,y)|^2 dt\, dx = ||v(y)||^2$$

$$\int_{-\infty}^{\infty} |v(t,x,y)|^2 dx = ||v(t,y)||^2 \quad .$$

Since $f,g \varepsilon B(x,\omega_o)$ it follows immediately from Lemma 1, that the solution $U(\tau,\omega,y)$ to (6),(7) belongs to $\mathcal{B}(\omega,\omega_o h)$. In fact from the estimate (8) we have that

$$||U(y)|| \leq A \left\{\alpha||F||+\beta||G|| \; e^{\Lambda(\omega_o,h)}\right.$$

for all τ,ω, $0 \leq y \leq h$, where α and β are given by

$$\alpha = \left\{[a(0)]^{1/4} + \tfrac{1}{4}[a(0)]^{-3/4}a'(0)\right\}$$
$$\beta = \left\{ a(0)\right\}^{-1/4}$$

and $\Lambda(\omega_o,h) = h^2E^2(C\omega_o+D)$. Hence from the Plancherel Theorem we have

$$(17) \qquad ||u(y)|| \leq A \left\{\alpha||f||+\beta||g||\right\}e^{\Lambda(\omega_o,h)}$$

in $R(h)$, so that $u(t,x,y) \varepsilon B(x,\omega_o,h)$. This means in particular that $u \varepsilon C^2(t,x)$. It follows from the classical theorems

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Thus all derivatives of u with respect to x depend Lipschitz continuously on the data, and continuous dependence in the L_2^x norm always implies continuous dependence in the maximum norm over x.

of ordinary differential equations that $U(\tau,\omega,y) \in C^2(y)$. Hence we have also $u(t,x,y) \in C^2(y)$. It is clear that $u(t,x,y)$ defined by (16) satisfies the differential equation (1) and the initial conditions (2), since $U(\tau,\omega,y)$ satisfies (6),(7).

Part (b) of the theorem follows from (17). In particular we see from this expression that the dependence of the solution u on the initial data f,g is of the type described by (3) with $M = A(\alpha+\beta)e^{\Lambda(\omega_o,h)}$ and $\theta = 1$.

The uniqueness of the solution $u(t,x,y)$ is deduced from the theory of ordinary differential equations (which provide a unique $U(\tau,\omega,y)$) and the Plancherel Theorem (which then provides $u(t,x,y)$). Since it follows from (8) that any $L_2^{\tau,\omega}$ solution of (6), (7) belongs to $\mathcal{B}(\omega,\omega_o,h) \cap C^2(y)$ for $0 \leq y \leq h$, we have in fact that our solution $u(t,x,y)$ is unique in the class $L_2^{t,x}$ for $0 \leq y \leq h$.

As mentioned previously our methods are also applicable to more general problems than (1),(2). In fact it is easy to prove the following theorem by techniques completely analogous to those employed above.

<u>Theorem 2</u>: Consider the ultrahyperbolic equation

$$(18) \qquad \sum_{i=1}^{m} a_i(y) u_{t_i t_i} = \sum_{j=1}^{n} b_j(y) u_{x_j x_j} + u_{yy}$$

in the region

$$\mathcal{R}(h) = \Big\{ (t_1,\ldots,t_m,x_1,\ldots,x_n,y) : -\infty < x_j, t_i < \infty, \; i=1,\ldots,m;\; j=1,\ldots,n;\; 0 \leq y \leq h \Big\},$$

where m,n are integers satisfying $m \geq 1$, $n \geq 1$, $a_i(y) \in C^2$, $b_j(y) \in C^0$ and $a_i > 0$ for all i and j. We suppose that the initial data

$$(19)\quad \begin{aligned} u(t_1,\ldots,t_m,x_1,\ldots,x_n,0) &= f(t_1,\ldots,t_m,x_1,\ldots,x_n) \\ u_y(t_1,\ldots,t_m,x_1,\ldots,x_n,0) &= g(t_1,\ldots,t_m,x_1,\ldots,x_n) \end{aligned}$$

are given on the timelike manifold $y = 0$ and satisfy the conditions

(a) $f,g \in L_2$ with respect to all t and x variables in $\mathcal{R}(h)$.

(b) $\overline{F}(t_1,\ldots,t_m,\omega_1,\ldots,\omega_n)$ and $\overline{G}(t_1,\ldots,t_m,\omega_1,\ldots,\omega_n)$ have support contained in the interval $\Omega_o \equiv \left\{ |\omega_j| \leq \omega_o,\ j=1,\ldots,n \right\}$.

(c) $\tau_i^2\ F(\tau_1,\ldots,\tau_m,\omega_1,\ldots,\omega_n)$ and $\tau_i^2\ G(\tau_1,\ldots,\tau_m,\omega_1,\ldots,\omega_n)$ $\in L_2$ with respect to all τ variables for all $(\omega_1,\ldots,\omega_n) \in \Omega_o$.

Then there exists a unique C^2 solution $u(t_1,\ldots t_m,x_1,\ldots,x_n,y)$ in $\mathcal{R}(h)$ having the properties

(a) $u \in L_2$ with respect to all t and x variables for $0 \leq y \leq h$, and the Fourier transform $\overline{U} = \overline{U}(t_1,\ldots,t_m,\omega_1,\ldots,\omega_m,y)$ has support in the interval Ω_o.

(b) u depends Lipschitz continuously on f and g in the L_2 norm with respect to the t and x variables for all y satisfying $0 \leq y \leq h$.

(c) u is unique in the class of C^2 functions of $t_1, \ldots, t_m, x_1, \ldots, x_n, y$ which are in L_2 with respect to the t and x variables in $\mathcal{R}(h)$.

IV. A MORE REFINED VERSION OF THEOREM 1

It seems reasonable to try and eliminate the requirement that $f, g \in L_2^t$, since the Fourier transforms of f and g with respect to t do not play an important role in the results of the previous sections. This requires some changes in the methods which lead to Theorem 1. In particular equation (1) is only partially reduced (by taking Fourier transforms with respect to x only) and one obtains the partial differential equation

$$\overline{U}_{yy}(t,\omega,y) = a\overline{U}_{tt}(t,\omega,y) + b\omega^2\overline{U}(t,\omega,y) \tag{20}$$

instead of the fully reduced equation (6). Then in place of the elementary Lemma 1 it is necessary to obtain estimates of the growth of the solution to (20) as a function of ω over some appropriate region.

In this section we shall state a theorem which is similar to Theorem 1, but which does not require that $f, g \in L_2^t$. We will not go through the proof here since it is similar to the arguments already given. The estimates of the growth of the solution to (20) which provides the basis for the proof are obtained directly from the integral equation methods for treating hyperbolic systems due to Courant and Lax.[3]

A few definitions are needed before the theorem can be stated:

(a) Define the region $R(\mu,\delta) \subset \mathbb{R}^3$ as the set of points (t,x,y) satisfying: $|t| \leq \mu$, $-\infty < x < \infty$, and $0 \leq y \leq \delta$ (μ and δ depend on the coefficients a,b, and the initial data f,g).

(b) For any function $h(t,x,y)$ defined in $R(\mu,\delta)$ we say $h \in C^2_{\text{Lip}}(t,x,\ldots)$ if h has continuous second derivatives with respect to $t,x,\ldots$ which satisfy a Lipschitz condition in $R(\mu,\delta)$.

(c) We define the "mixed norm" $|||h(y)|||$ for any function $h(t,x,y) \in B(x,\omega_o) \cap C^2_{\text{Lip}}(t)$ in $R(\mu,\delta)$ by

$$|||h(y)||| = \sup_{|t|\leq\mu} ||h(t,y)|| \quad ,$$

where as in Section III

$$||h(t,y)|| = \int_{-\infty}^{\infty} |h(t,x,y)|^2 dx.$$

Now consider the equation

$$a(t,y)u_{tt}(t,x,y) = b(t,y)u_{xx}(t,x,y) + u_{yy}(t,x,y) \tag{21}$$

with timelike initial data

$$\begin{aligned} u(t,x,y)|_{y=0} &= f(t,x) \\ u_y(t,x,y)|_{y=0} &= g(t,x) \end{aligned} \tag{22}$$

in the region $R(\mu,\delta)$. We shall assume that

(a) $f(t,x)$, $g(t,x)$, $\frac{\partial f(t,x)}{\partial t}$ ε $B^*(x,\omega_o)$ in $R(\mu,\delta)$, where the conditions defining the classes B^* and B (see Section II) are the same except that membership in L_2^t is no longer required for B^* functions.

(b) $\overline{F}(t,\omega)$, $\overline{G}(t,\omega)$, $\frac{\partial \overline{F}(t,\omega)}{\partial t}$ ε $C^2_{Lip}(t)$ for $|\omega| \leq \omega_o$ and $|t| \leq \mu$.

(c) $a(t,y) > 0$; $a(t,y)$, $b(t,y)$ ε $C^2_{Lip}(t,y)$ in $R(\mu,\delta)$.

We can now state

<u>Theorem 3:</u> If the initial data f,g satisfy (a), (b) and the coefficients a,b satisfy (c), then for sufficiently small δ there exists a unique solution $u(t,x,y)$ to (21), (22) in $R(\mu,\delta)$ with the following properties:

(a) $u(t,x,y)$ and all its first partial derivatives belong to $B^*(x,\omega_o)$ in $R(\mu,\delta)$.

(b) u,u_t,u_y ε $C^2_{Lip}(t,y)$ in $R(\mu,\delta)$. u is an entire function of x of exponential type ω_o.

(c) u and all its first partial derivatives depend Lipschitz continuously in the

$||| \quad |||$ norm on the data $f,g,\frac{\partial f}{\partial t}$.*

It is important to emphasize that Theorem 3 provides results which are local in y(i.e., δ is a suitable "small" parameter). The results, however, are not local in t (i.e., μ need not be small). In fact, if $f,g,\frac{\partial f}{\partial t}$, a and b are all bounded as functions of t (in addition to their other assumed properties),

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* Actually more can be said concerning the continuous dependence of u and its x-derivatives as functions of x - see the footnote to Theorem 1.

then we may take $\mu = \infty$. Even when the data is not bounded, μ may be taken as large as desired, but in general this will necessitate smaller δ. We also point out that $R(\mu,\delta)$ may be replaced by the more general region

$$R(\mu,\delta,t_o) = \left\{(t,x,y) : |t-t_o| \leq \mu,\ -\infty < x < \infty,\ 0 \leq y \leq \delta\right\}.$$

The relaxation of the requirement that $f,g \in L_2^t$ has an important consequence: Theorem 3 is applicable to timelike problems in which the coefficients of the differential equation depend on t as well as y (compare equations (1) and (21)).

The methods used to prove Theorem 3 do not generalize to the ultrahyperbolic problem (18), (19).

It is a pleasure to acknowledge several helpful discussions with Professor Fritz John which contributed significantly to the formulation and study of the problems discussed in this report.

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J. M. Zimmerman
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